The category of finite dimensional operator spaces

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Two views on quantum operations

Schrödinger view:

$\mathcal{S} := \begin{cases} \text{ob: finite dimensional Hilbert spaces} \\ \text{mor: CPTP maps } \varphi : T(H) \to T(K) \end{cases}$

Maps represents operations on states

Heisenberg view:

$\mathcal{H} := \begin{cases} \text{ob: finite dimensional Hilbert spaces} \\ \text{mor: NCPU maps } \varphi : B(H) \to B(K) \end{cases}$

Maps represents operations on effects

Heisenberg-Schrödinger duality

Given two Hilbert spaces H_1 and H_2 we have an isomorphism

$$\mathbf{CPTP}(T(H_1), T(H_2)) \cong \mathbf{NCPU}(B(H_2), B(H_1))$$

This gives an equivalence of categories! $\mathcal{S} \cong \mathcal{H}^{\text{op}}$

Remark

We want one category - The keyword is completely!

Why care... pt. 2

The operator space setting

- Operator space theory - non-commutative (quantum) Banach space theory
  - Banach spaces are nice categorically $\implies$ Operator spaces are nice?
- Rich theory with many interesting constructions (tensor products etc)
  - including one non-commutative and self-dual tensor $((A \otimes_h B)^* = A^* \otimes_h B^*)$ for A and B f.d

$B(H)$ and $T(H)$ are operator spaces!

- We also have that $B(H) \cong T(H)^*$ as operator spaces
- $T(H_1) \widehat{\otimes} T(H_2) \cong T(H_1 \otimes H_2)$
- $B(H_1) \check{\otimes} B(H_2) \cong B(H_1 \otimes H_2)$

Operator Spaces

Definition (Operator Space)

A *operator space* is a closed linear subspace of $A \subset B(H)$ for some Hilbert space H.

Properties

- $M_n(A) \subset M_n(B(H)) \cong B(H^n)$ *determines a norm on* $M_n(A)$.
- *Main result: fully abstract definition - we do not need to point out* H

Examples

- $\mathbb{C} \cong B(\mathbb{C})$
- $B(H)$ - Space of bounded operators
- $T(H)$ - Space of trace class operators
- H_c, given by $H \cong B(\mathbb{C}, H)$ gives O.S structure on any Hilbert space H

Operator Spaces, Effros and Ruan 2000

Maps between operator spaces

Definition (Amplification)

Let A, B be operator spaces, then the nth *amplification* of a linear map $\varphi : A \to B$ is given by

$$\begin{array}{rcl} \varphi_n : M_n(A) & \to & M_n(B) \\ a & \mapsto & [\varphi(a_{i,j})] \end{array}$$

Definition (completely bounded-ness)

Let $\varphi : A \to B$ be a linear map. We define the *cb-norm* on such a morphism by

$$\|\varphi\|_{\mathrm{cb}} = \sup\{\|[\varphi(a_{i,j})]\|_n \mid n \in \mathbb{N},\ a = [a_{i,j}] \in M_n(A),\ \|[a_{i,j}]\|_n \leq 1\}$$

- *completely bounded* (c.b) if $\|\varphi\|_{\mathrm{cb}} < \infty$
- *completely contractive* (c.c) if $\|\varphi\|_{\mathrm{cb}} \leq 1$

The category in question

Definition

Let **fdOS** denote the category

- objects: finite dimensional operator spaces
- morphisms: completely contractive maps

Properties

- *Model of MALL*
  - *(We focus on the MLL-part here)*
- *BV-category*

Classical MLL

Definition (Multiplicative Linear Logic)

$$A ::= X \mid A^* \mid I \mid A \otimes A' \mid \bot \mid A \parr A'$$

$$A \multimap A' := A^* \parr A'$$

Coherences

- Associativity and Commutativity of $\otimes$ and $\parr$
- De Morgan: $(A \otimes B)^* = A^* \parr B^* \quad (A \parr B)^* = A^* \otimes B^*$
- Unit : $I \otimes A = A \otimes I = A \quad \bot \parr A = A \parr \bot = A$
- Double dual: $A^{**} = A$

Inference rules

$$\frac{}{\vdash A^{\bot}, A} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

Models of classical MLL

Definition ($*$-autonomous category)

A symmetric monoidal closed category $(\mathcal{C}, I, \otimes, \multimap)$ is called $*$-*autonomous* if the transpose $\partial_A : A \to ((A \multimap \bot) \multimap \bot)$ of the evaluation map $\mathrm{eval}_A : A \otimes (A \multimap \bot) \to \bot$ is an isomorphism.

Proposition

Any $*$-autonomous category is a model of classical MLL where

$$
\begin{aligned}
[\![A^*]\!] &:= [\![A]\!] \multimap \bot & \qquad [\![I]\!] &:= I \\
[\![A \otimes B]\!] &:= [\![A]\!] \otimes [\![B]\!] & \qquad [\![\bot]\!] &:= \bot \\
[\![A \parr B]\!] &:= [\![A]\!]^* \multimap [\![B]\!] & &
\end{aligned}
$$

Coherences and inferences are isomorphisms and morphisms resp.

Remark

Coherences and inferences are actually natural!

Constructions on operator spaces

Mapping spaces

- CB-space: $\mathcal{CB}(A, B)$ (the space of completely bounded maps)
  - Norm given by $\| - \|_{\mathrm{cb}}$
- Dual space: $A^* := \mathcal{CB}(A, \mathbb{C})$

Properties

- *Functorial*
- *UP:* $\mathcal{CB}(A, \mathcal{CB}(B, C)) \cong \mathcal{JCB}(A, B; C)$
- $A \cong A^{**}$ *when* A *is f.d*

Remark

$\mathcal{JCB}(A, B; C)$ *is the space of jointly completely bounded maps.*

Operator Spaces, Effros and Ruan 2000

Constructions on operator spaces pt 2

Tensors

- Projective tensor: $A \widehat{\otimes} B$
  - UP: $\mathcal{CB}(A \widehat{\otimes} B, C) \cong \mathcal{JCB}(A, B; C) \quad (\cong \mathcal{CB}(A, \mathcal{CB}(B, C)))$
  - Maximal cross matrix tensor (mapping out property)
- Injective tensor: $A \check{\otimes} B$
  - norm induced by $A \otimes B \hookrightarrow \mathcal{CB}(A^*, B)$
  - Minimal cross matrix tensor (mapping in property)

Properties

- *Functorial wrt c.b and c.c*
- *Symmetric:* $A \widehat{\otimes} B \cong B \widehat{\otimes} A \qquad A \check{\otimes} B \cong B \check{\otimes} A$
- *Associative:*
  $(A \widehat{\otimes} B) \widehat{\otimes} C \cong A \widehat{\otimes} (B \widehat{\otimes} C) \qquad (A \check{\otimes} B) \check{\otimes} C \cong A \check{\otimes} (B \check{\otimes} C)$
- *Has unitors:* $\mathbb{C} \widehat{\otimes} A \cong A \cong A \widehat{\otimes} \mathbb{C} \qquad \mathbb{C} \check{\otimes} A \cong A \cong A \check{\otimes} \mathbb{C}$

fdOS as a model of classical MLL

Proposition

fdOS *equipped with the projective tensor product and the CB-space,* $(\mathbf{fdOS}, \mathbb{C}, \widehat{\otimes}, \mathcal{CB}(-,-))$*, is monoidal closed.*

fdOS is a model of MLL

$*$-autonomous with

- multiplicative conjunction $X \otimes Y := X \,\widehat{\otimes}\, Y$
- unit of multiplicative conjunction $I := \mathbb{C}$
- multiplicative disjunction $X \parr Y := X \,\check{\otimes}\, Y \quad (\cong \mathcal{CB}(X^*, Y))$
- unit of multiplicative disjunction $\perp := \mathbb{C}^* \cong \mathbb{C}$
- dual $X^* := \mathcal{CB}(X, \mathbb{C})$ such that $X \cong X^{**}$

MALL and Models of MALL

Definition

$$A ::= X \mid A^* \mid 1 \mid A \otimes A' \mid \bot \mid A \parr A' \mid 0 \mid A \oplus A' \mid \top \mid A \& A'$$

Coherences

- Associativity and Commutativity of $\oplus$ and $\&$
- De morgan: $(A \oplus B)^* = A^* \& B^* \quad (A \& B)^* = A^* \oplus B^*$
- Unit : $0 \oplus A = A \oplus 0 = A \quad \bot \parr A = A \parr \bot = A$

Proposition

Any cartesian $*$-autonomous category (hence also cocartesian) is a model of classical MALL logic where

$$[\![A \& B]\!] := [\![A]\!] \times [\![B]\!]$$

$$[\![A \oplus B]\!] := [\![A]\!] + [\![B]\!]$$

Products and coproducts in **fdOS**

Definition (direct sums)

Let A and B be operator spaces, the ∞-*direct sum* and 1-*direct sum*

- underlying vector space the cartesian product, $A \times B$
- $\|(a,b)\|_\infty := \max(\|a\|, \|b\|)$. The operator space structure is given by

$$M_n(A \oplus^\infty B) \cong M_n(A) \oplus^\infty M_n(B)$$

- $\|(a,b)\|_1$ is defined using

$$\begin{array}{rcl} A \oplus^1 B & \to & (A^* \oplus^\infty B^*)^* \\ (a,b) & \mapsto & ((f,g) \mapsto f(a) + g(b)) \end{array}$$

Proposition

We have that $\oplus^\infty$ *is the product and* $\oplus^1$ *is the coproduct in* **fdOS**

BV-logic: an extension of classical MLL

Definition

$$S ::= I \mid X \mid S^* \mid S \otimes S \mid S \parr S \mid S \triangleright S$$

- Associativity of $\triangleright$
- Negation: $(A \triangleright B)^* = A^* \triangleright B^* \quad I^* = I$
- Commutativity of $\triangleright$ is not required

Inference rules

$$C \to C\langle I \rangle \quad C\langle (A \triangleright B) \otimes (C \triangleright D) \rangle \to C\langle (A \otimes C) \triangleright (B \otimes D) \rangle$$

$$C\langle I \rangle \to C\langle A \parr A^* \rangle \quad C\langle (A \parr B) \otimes C \rangle \to C\langle (A \otimes C) \parr B \rangle$$

Remark

Deep inference $\implies$ Inference rules/coherences are natural maps?

Guglielmi 1999

BV-category

Definition (BV-category with negation)

A $*$-*autonomous* category $(\mathcal{C}, I, \otimes, \multimap)$ with an additional monoidal structure $(\mathcal{C}, J, \triangleright)$ that is *normal duoidal* to $(\mathcal{C}, I, \otimes)$.

Remark

- $w : (A \triangleright B) \otimes (C \triangleright D) \to (A \otimes C) \triangleright (B \otimes D)$
- $I \cong J$
- *(Self-duality can be derived from these premises)*

What is $\triangleright$ in fdOS?

We need yet another tensor, more is more!

Blute, Panangaden, and Slavnov 2010

Haagerup tensor product, finite dimensional case

Definition (matrix inner product)

The *matrix inner product* of $a \in \mathbb{M}_{n,r}(A)$ and $b \in \mathbb{M}_{r,m}(B)$ is

$$[(a \odot b)_{i,j}] = [\sum_{k=1}^{r} a_{i,k} \otimes b_{k,j}] \in \mathbb{M}_{n,m}(A \otimes B)$$

Definition (Haagerup norm)

- The *Haagerup norm* of a $v \in \mathbb{M}_n(A \otimes B)$ is

$$\|v\|_h = \inf\{\|a\|_{n,r}\|b\|_{r,n} \mid a \odot b = v\}$$

- The *Haagerup tensor product*, denoted $A \otimes_h B$, has as underlying vector space $A \otimes B$ and matrix norm $\| - \|_h$.

Operator Spaces, Effros and Ruan 2000

Properties of the Haagerup tensor

Proposition (U.P of the Haagerup tensor)

Let $\varphi : A \otimes_h B \to B(H, K)$ *be a linear map, then TFAE*

- φ *is c.b (c.c)*
- *there are c.b (c.c) maps* $\varphi_1 : A \to B(H, L)$ *and* $\varphi_2 : B \to B(L, K)$ *s.t*

$$\varphi(a \otimes b) = \varphi_1(a) \circ \varphi_2(b)$$

Properties

- *Let* A *and* B *be two f.d operator spaces we have*

$$A^* \otimes_h B^* \cong (A \otimes_h B)^*$$

- *The Haagerup tensor is non-symmetric*
  - *Pf. Given a finite dimensional Hilbert space* H *we have isomorphisms*

$$H_c \otimes_h (H_c)^* \cong B(H) \qquad (H_c)^* \otimes_h H_c \cong T(H)$$

fdOS is a BV-category

Haagerup tensor

- Functorial wrt c.b and c.c maps
- Associative
- Unitors: $\mathbb{C} \otimes_h A \cong A \cong A \otimes_h \mathbb{C}$

NB: These make $(\mathbf{fdOS}, \mathbb{C}, \otimes_h)$ into a monoidal category

fdOS is a BV-category with negation

- $*$-autonomous
- Duoidality for free, as $\otimes_h$ is functorial wrt completely bounded maps

$$w^* := (A \check{\otimes} B) \otimes_h (C \check{\otimes} B) \cong \mathcal{CB}(A^*, B) \otimes_h \mathcal{CB}(C^*, D) \longrightarrow \mathcal{CB}(A^* \otimes_h C^*, B \otimes_h D)$$

$$\cong \mathcal{CB}((A \otimes_h C)^*, B \otimes_h D) \cong (A \otimes_h c) \check{\otimes} (B \otimes_h D)$$

- Has the same unit as $\widehat{\otimes}$

What do we have?

Organizing $\mathcal{H}$ and $\mathcal{S}$ in the same category

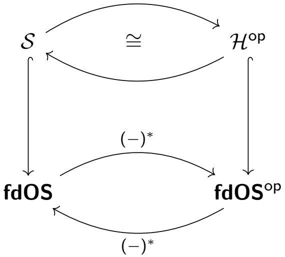
We have non-full embeddings of $\mathcal{H}$ and $\mathcal{S}$ into **fdOS**.

$$\mathcal{S} \xrightarrow{T(-)} \mathbf{fdOS} \xleftarrow{B(-)} \mathcal{H}$$

- We have a dualizing functor

$$(-)^* : \mathbf{fdOS}^{\mathrm{op}} \to \mathbf{fdOS}$$

- This duality restricts along the inclusions $\mathcal{H}$ and $\mathcal{S}$ to the equivalence $\mathcal{S} \cong \mathcal{H}^{\mathrm{op}}$



Limitations

The embeddings of $\mathcal{H}$ and $\mathcal{S}$ are not full, consequently there are morphisms in **fdOS** that do not correspond to quantum operations

What do we want? - a PhD!

Build a "category of operator spaces" E such that

1. There are *full* and faithful embeddings of $\mathcal{H}$ and $\mathcal{S}$
2. Images of $\mathcal{H}$ and $\mathcal{S}$ are dual
3. E "inherits" the MALL-structure from **fdOS**
4. Preserve the non-commutative structure

Small sketch

- Double gluing [Hyland and Schalk 2003]
- Homset-double gluing is not sufficient! (see 4. above)
- The new category
  - (A, S) with A an operator space
  - $f : (A, S) \to (B, R)$ s.t $f : A \to B$ and $f^*S \hookrightarrow R$
- S needs to be closed
- Choosing the right closedness conditions is non-trivial

PhD project supervised by: Vladimir Zamdzhiev and Benoît Valiron

Summary: **fdOS** is...

... a model of MLL

$*$-autonomous with

- dual $X^* := \mathcal{CB}(X, \mathbb{C})$ such that $X \cong X^{**}$
- multiplicative conjunction $X \otimes Y := X \widehat{\otimes} Y$ (the projective tensor)
- multiplicative disjunction $X \parr Y := X \check{\otimes} Y$ (injective tensor)
- unit $I := \mathbb{C}$

... a BV-category

- dual $X^* := \mathcal{CB}(X, \mathbb{C})$ such that $X \cong X^{**}$
- multiplicative conjunction $X \otimes Y := X \widehat{\otimes} Y$ (the projective tensor)
- multiplicative disjunction $X \parr Y := X \check{\otimes} Y$ (injective tensor)
- sequential $X \triangleright Y := X \otimes_h Y$
- unit $I := \mathbb{C}$