The Category of Finite Dimensional Operator Spaces

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M2 Internship report - Logique Mathématique et Fondements de l'Informatique

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Abstract. In this report we investigate the semantical properties of the category of finite dimensional operator spaces, in particular we prove that it is a non-degenerate model of multiplicative additive linear logic. We also argue that this category gives a good basis for modeling BV-logic, an extension of multiplicative linear logic by showing that it is a BV-category.

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1 Introduction

Many concepts in quantum information theory can be modeled using mathematical methods from linear algebra, functional analysis and operator algebras. While these are often well understood from a physics and mathematics perspective, they are not always well understood from a computer science perspective. The *Heisenberg-Schrödinger duality* describes the equivalence between two views on the evolution of quantum systems. Mathematically it is stated as an equivalence of two types of maps between two types of *operator spaces* that are dual to each other under the operator space dual, see [ER00]. The categorical properties of this duality suggest that the Heisenberg-Schrödinger duality could be seen from a linear logic (LL) point-of-view. Indeed the negation in LL is a type of duality. Traditionally the basis for quantum programming languages has been intuitionistic LL [SV05], however, to properly capture the Heisenberg-Schrödinger duality, we need double negation and consequently classical LL. Thus a first step towards a computational or logical characterization of the Heisenberg-Schrödinger duality would be to prove that the category of operator spaces gives semantics for some fragment of classical LL.

The theory of operator spaces has been extensively studied from a functional analysis perspective, consequently the basis theory is very rich, [ER00]. A significant part of it parallels Banach space theory, which has interesting categorical properties. For example, the category of finite dimensional Banach spaces is known to be a model of classical multiplicative additive linear logic (MALL), [Egg14]. Just as for Banach spaces, many constructions and results concerning operator spaces translate well into categorical concepts. For example, we can prove that the operator space dual gives a dualizing endofunctor on category of finite dimensional operator spaces, such that a space is naturally isomorphic to its double dual.

Our main objective in this work will thus be to investigate the semantical properties of the category of finite dimensional operator spaces. In particular we prove that it is a non degenerate model of (classical) MALL, where connectives can be interpreted by notions inherent to functional analysis. As we can define a non-commutative and self dual tensor on finite dimensional operator space, we also argue that this category might provide a good basis for modeling *BV-logic*, an extension of the multiplicative fragment of LL. To this end, we prove that the category of finite dimensional operator spaces is a *BV-category*, in the sense of [BPS10]. While this is not sufficient to claim that it is a model of BV-logic, it is not yet clear what a categorical model of BV-logic should be.

We assume that the reader has basic background in category theory as well as linear logic. We do however provide a introduction to the necessary concepts in operator spaces theory, with a slight emphasis on the categorical properties of the constructions, assuming only basic linear algebra.

2 Operator spaces

Recall the well known result stating that any normed space can be realized as a *function space*, a space of bounded linear endofunctions with the uniform norm. The study of operator spaces arises when we consider the linear subspaces of bounded operators on a Hilbert space. This space not only inherits an operator norm, as any $n \times n$ matrix of operators on a Hilbert space can be seen as an operator on H^n, it inherits an operator norm on each of the spaces of matrices with entries in the subspace. Because of this higher structure, operator spaces can be seen as the "non-commutative" or "quantized" generalization of the theory of Banach spaces. The definitions and results in this section are standard in operator space theory and can be found in [ER00], [BP91] or [Ble16], we thus only give a handful of proofs that either illuminates the theory or illustrates the usefulness of a result.

2.1 Preliminaries

In order to define operator spaces we first need to introduce some basic concepts and notations from functional analysis. As mentioned above, the notion of operator space rely on the structure of two other types of normed spaces.

Definition 2.1. A *Banach space* is a normed vector space that is complete with respect to the metric induced by the norm.

Proposition 2.2. *Any finite dimensional normed space is a Banach space.*

Definition 2.3. A *Hilbert space* is an inner product space that is also complete with respect to the metric induced by the inner product.

As the metric induced by the inner product on a Hilbert space also induces a norm, every Hilbert space is also has a Banach space structure.

We can naturally define a norm on linear maps between normed spaces in the following way.

Definition 2.4. Given a linear map between normed vector spaces $\varphi : A \to B$ we define the *operator-norm* of φ by

$$\|\varphi\|_{\mathrm{op}} = \sup\{\|\varphi(a)\| \mid \|a\| \leq 1\}$$

We say that a linear map between normed spaces φ is *bounded* if $\|\varphi\| < \infty$.

A simple example of a Banach space is the vector space of bounded linear functions between two Hilbert spaces with the operator norm. For Hilbert spaces H and K this space is denoted by $\boldsymbol{B}(H, K)$, and if $H = K$ the notation is simplified to $\boldsymbol{B}(H)$. Note that Banach spaces of the form $\mathbf{B}(H)$ are important as they enable our first definition of operator spaces.

Definition 2.5. Given a vector space V, the $n \times m$ *matrix space* of V is the vector space of $n \times m$ matrices with entries in V, denoted $\mathbb{M}_{n,m}(V)$.

To simplify notation we denote $\mathbb{M}_{n,n}(V)$ by $\mathbb{M}_n(V)$ and $\mathbb{M}_n(\mathbb{C})$ by $\mathbb{M}_n$. Another notation we introduce is setting $H^n = H \oplus ... \oplus H$, where $\oplus$ is the direct sum of Hilbert spaces. We have an isometric identification $\mathbb{M}_n(\mathbf{B}(H)) = \mathbf{B}(H^n)$, which in particular implies that each matrix space of $\mathbf{B}(H)$ can be seen as operator normed spaces.

Definition 2.6. [ER00, p. 20] A *concrete operator space* A on a Hilbert space H, is a complete linear subspace $A \hookrightarrow \mathbf{B}(H)$.

This naturally induces a norm on each matrix space of A given by the inclusion

$$\mathbb{M}_n(A) \hookrightarrow \mathbb{M}_n(\mathbf{B}(H)) = \mathbf{B}(H^n)$$

Example 2.7. Consider the isometry $\iota : \mathbb{C} \cong \mathbf{B}(\mathbb{C}) : c \mapsto (a \mapsto ca)$, this induces an operator space structure on $\mathbb{C}$, to be specific

$$\begin{array}{rcccc} \mathbb{M}_n(\mathbb{C}) & \cong & \mathbb{M}_n(\mathbf{B}(\mathbb{C})) & = & \mathbf{B}(\mathbb{C}^n) \\ c = [c_{i,j}] & & \longmapsto & & [\iota(c_{i,j})] \end{array}$$

We denote by $M_{n,m}$ the normed space that have $\mathbb{M}_{n,m}$ as underlying vector space equipped with the norm given by the isometric identification $\mathbb{M}_{n,m} = \mathbf{B}(\mathbb{C}^m, \mathbb{C}^n)$.

Having to point out the Hilbert space, can make it hard to verify that new constructions on operator spaces, indeed give other operator spaces. In order to have a completely abstract definition we can instead require two conditions to be satisfied by the norms on the matrix spaces, these guarantee that the spaces in question behave as if they were embedded into a $\mathbf{B}(H)$, see [ER00, Ch. 2].

Definition 2.8. [ER00, p. 20] A *matrix norm* on a linear space A is an assignment of a norm $\| - \|_n$ on $\mathbb{M}_n(A)$ for each $n \in \mathbb{N}$.

Given such a space A, the normed space $(\mathbb{M}_n(A), \| - \|_n)$ will be denoted by $M_n(A)$.

Definition 2.9. [ER00, p. 20] An *abstract operator space* A is a linear space equipped with a matrix norm, that is a Banach space with respect to $\| - \|_1$, and such that

- **(M1)** $\|a \oplus b\|_{m+n} = \max\{\|a\|_m, \|b\|_n\}$
- **(M2)** $\|\alpha a \beta\|_n \leq \|\alpha\| \|a\|_m \|\beta\|$

for $a \in M_m(A)$, $b \in M_n(A)$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$.

The two axioms above are usually referred to as Ruan's axioms. As a consequence to Proposition 2.2 any matrix normed finite dimensional space satisfying Ruan's axioms is an operator space.

Example 2.10. Let H be a Hilbert space, then the *column Hilbert operator space* H_c is H with the operator space matrix norm given by the isometry $C : H \to \mathbf{B}(\mathbb{C}, H) : h \mapsto (n \mapsto nh)$. In particular, given an $h \in M_n(H)$ we have a

$$\begin{array}{rcl} C_n(h) : \mathbb{C}^n & \to & H^n \\ \big[c_i\big] & \mapsto & \big[C(c_i)\big] \end{array}$$

and $\|h\|_c = \|C_n(h)\|$. In particular, we have that $\mathbb{C}_c = \mathbb{C}$.

As the important part of the structure of operator spaces is given by the matrix space structure, morphisms between operator spaces naturally need to reflect this.

Definition 2.11. [ER00, sec 2.2] The *nth amplification* of a map between two abstract operator spaces $\varphi : A \to B$ is defined by the following

$$\begin{array}{rcl} \varphi_n : M_n(A) & \to & M_n(B) \\ a & \mapsto & [\varphi(a_{i,j})] \end{array}$$

Now we can define the analogous notion of a bounded linear function for operator spaces:

Definition 2.12. [ER00, sec 2.2] Let A and B be operator spaces and $\varphi : A \to B$ be a linear map. We define the *cb-norm* on a morphism between operator spaces by

$$\|\varphi\|_{\text{cb}} = \sup\{\|[\varphi(a_{i,j})]\|_n \mid n \in \mathbb{N},\ a = [a_{i,j}] \in M_n(A),\ \|[a_{i,j}]\|_n \leq 1\}$$

Definition 2.13. [ER00, sec 2.2] Let A and B be operator spaces and $\varphi : A \to B$ be a linear map, we call φ *completely bounded* if $\|\varphi\|_{\text{cb}} < \infty$.

As well as the analogs of contractions and isometries:

Definition 2.14. [ER00, sec 2.2] Let A and B be operator spaces and $\varphi : A \to B$ be a linear map, we call φ *completely contractive* if $\|\varphi_n(a)\|_n \leq \|a\|_n$ for each $a \in M_n(A)$.

(This is equivalent to $\|\varphi\|_{\mathrm{cb}} \leq 1$).

Definition 2.15. [ER00, sec 2.2] Let A and B be operator spaces and $\varphi : A \to B$ be a linear map, we call φ *completely isometric* if $\|\varphi_n(a)\|_n = \|a\|_n$ for each $a \in M_n(A)$.

Similarly to how the operator norm gives the mapping spaces of Banach spaces a Banach space structure, the cb-norm induces an operator space structure on the linear space of completely bounded linear functions between two operator spaces. Given two operator spaces A and B, the linear space of completely bounded maps from A to B together with the cb-norm is denoted by $\mathbf{CB}(A, B)$.

Proposition 2.16. *[ER00, Prop. 3.2.5] Given operator spaces A and B, $\boldsymbol{CB}(A, B)$ is a Banach space.*

The matrix norm of this mapping space is induced by the following isometric identification.

Proposition 2.17. *[Ble16, Par. 1.2.18] We have an isometric isomorphism $M_n(\boldsymbol{CB}(A, B)) \cong \boldsymbol{CB}(A, M_n(B))$*

Proof. We define a linear function from $M_n(\mathbf{CB}(A, B))$ to $\mathbf{CB}(A, M_n(B))$ by mapping a matrix of cb-maps

$$f = [f_{i,j}] \in M_n(\mathbf{CB}(A, B))$$

to the morphism from A to B that sends $a \in A$ to $[f_{i,j}(a)]$, which is linear and completely bounded as each of the $f_{i,j}$'s are. This is an isometry as

$$\|[f_{i,j}]\|_n = \sup\{\|[f_{i,j}(a_{k,l})]\|_{nm} \mid n \in \mathbb{N},\ [a_{k,l}] \in M_m(A),\ \|[a_{k,l}]\|_m \leq 1\} = \|(a \mapsto [f_{i,j}(a)])\|_{\mathrm{cb}}$$

□

We can now verify that the cb-norm gives this an operator space structure.

Proposition 2.18. *[Ble16, Par. 1.2.18] The linear space of completely bounded maps between two operator spaces, denoted $\mathcal{CB}(A, B)$, equipped with $\| - \|_{cb}$ is an operator space.*

Proof. By Proposition 2.16 the underlying normed space is a Banach spaces. No we verify **M1**, suppose $f = [f_{ij}] \in M_m(\mathcal{CB}(A, B))$ and $g = [g_{lk}] \in M_n(\mathcal{CB}(A, B))$, then

$$\begin{aligned} \|f \oplus g\|_{m+n} &= \left\| \begin{pmatrix} [f_{ij}] & 0 \\ 0 & [g_{lk}] \end{pmatrix} \right\| \\ &= \|f \oplus g\|_{cb} \\ &= \sup\{r \mid \|[(f \oplus g)_{ij}(x_{lk})]\| \leq r\|x\| \text{ with } x \in M_n(A)\} \\ &= \max(\|f\|_m, \|g\|_n) \end{aligned}$$

And for **M2**

$$\|\alpha[f_{ij}]\beta\| = \|[\sum_j \sum_k \alpha_{ij} f_{jk} \beta_{kl}]\| \leq \|\alpha\| \|f\|_{cb} \|\beta\|$$

□

More elaborate descriptions can be found in [ER00, Sec. 2.1-2.2]

By taking the codomain in this mapping space to be $\mathbb{C}$ we obtain a notion of dual space for operator spaces.

Definition 2.19. [ER00, (3.2.1)] Given an operator space A we define its (operator space) *dual* to be the operator space $A^* := \mathcal{CB}(A, \mathbb{C})$

As for any vector space we have a canonical inclusion of an operator space into its double dual.

Proposition 2.20. *[ER00, Prop. 3.2.1] Given an operator space A, the canonical inclusion*

$$\begin{array}{rcl} d_A : A & \hookrightarrow & A^{**} \\ a & \mapsto & (f \mapsto f(a)) \end{array}$$

is a natural complete isometry.

To prove this we make use of the following lemma:

Lemma 2.21. *[ER00, Lemma 2.3.4] Let A be an operator space, then for any $a \in M_n(A)$ there exists a complete contraction $f : A \to M_n$ such that*

$$\|f_n(a)\| = \|a\|$$

Proof of Proposition 2.20. To prove that this is a complete isometry, let $v \in M_n(A)$ and $f \in M_n(\mathcal{CB}(A, \mathbb{C}))$ then

$$((d_A)_n(v))_n(f) = [f_{k,l}(v_{i,j})] = f_n(v)$$

thus

$$\begin{array}{rcl} \|d_A(v)_n\|_{\text{cb}} & = & \sup\{\|((d_A)_n(v))_n(f)\| \mid f \in M_n(\mathcal{CB}(A, \mathbb{C})), \|f\|_{\text{cb}} \leq 1\} \\ & = & \sup\{\|f(v)\| \mid f \in M_n(\mathcal{CB}(A, \mathbb{C})), \|f\|_{\text{cb}} \leq 1\} \\ & = & \sup\{\|f_n(v)\| \mid f \in \mathcal{CB}(A, M_n), \|f\|_{\text{cb}} \leq 1\} \\ & = & \|v\| \end{array}$$

The last step follows from the lemma above. □

Corollary 2.22. *If A is a finite dimensional operator space, then the embedding d_A is an isometric isomorphism.*

Proof. As A and A^{**} have the same dimension and both are finite dimensional the embedding d_A must by the rank-nullity theorem of vector spaces be an isomorphism of vector spaces. □

We now present some useful results we will need later.

Proposition 2.23. *Given two completely bounded maps $f : A \to B$ and $g : B \to C$ then the cb-norm is submultiplicative, that is*

$$\|gf\|_{cb} \leq \|g\|_{cb}\|f\|_{cb}$$

Proof. We have

$$\begin{aligned} \|gf\|_{\mathrm{cb}} &= \sup\{\|gf\|_n \mid n \in \mathbb{N}\} \\ &= \sup\{\|gf(a)\|_n \mid n \in \mathbb{N}\ \|a\| \leq 1\} \\ &\leq \sup\{\|g\|_n\|f(a)\|_n \mid n \in \mathbb{N}\ \|a\| \leq 1\} \\ &\leq \sup\{\|g\|_n\|f\|_n \mid n \in \mathbb{N}\} \\ &\leq \sup\{\|g\|_n \mid n \in \mathbb{N}\} \times \sup\{\|f\|_n \mid n \in \mathbb{N}\} \\ &= \|g\|_{\mathrm{cb}}\|f\|_{\mathrm{cb}} \end{aligned}$$

□

Proposition 2.24. *(appears as a claim in [Ble16, Par. 1.2.3]) Let $\varphi : A \to B$ be an isomorphism of the underlying vector spaces, if both φ and φ^{-1} are completely contractive then φ is a completely isometric isomorphism.*

Proof. As φ is completely contractive, that is $\|\varphi_n(a)\|_n \leq \|a\|_n$ for all $n \in \mathbb{N}$ and $a \in M_n(A)$, it is sufficient to prove that $\|a\|_n \leq \|\varphi_n(a)\|_n$. We have that $\|\varphi_n^{-1}(b)\|_n \leq \|b\|_n$, as φ^{-1} is completely contractive, thus taking $b = \varphi(a)$ gives us $\|\varphi_n^{-1}(\varphi(a))\|_n = \|a\|_n \leq \|\varphi_n(a)\|_n$. □

The following proposition tells us that we can define a completely bounded function between operator spaces by specifying it on a dense subspace of the domain. This is useful as we often define constructions on operator spaces by completing with respect to a norm.

Proposition 2.25. *Let $A \subset B$ be a dense linear subspace of some operator space B with the inherited matrix norm, C be an operator space and $f : A \subset B \to C$ be a completely bounded map, then f extends uniquely to a map $\bar{f} : B \to C$ such that $\|\bar{f}\|_{cb} = \|f\|_{cb}$.*

Lemma 2.26. *[ER00, p. 22] Given an operator space A then and sequences a sequence of $n \times n$ matrices whose entries converge in A, then this sequence of matrices converge in $M_n(A)$.*

Proof. We have that

$$\|a\|_n = \|[a_{i,j}]\|_n = \|\sum_{i,j} a_{i,j} \otimes E_{i,j}\|_n \leq \sum_{i,j} \|E_{i,1}(a_{i,j} \otimes E_{i,j})E_{1,j}\| \leq \sum_{i,j} \|a_{i,j} \oplus \mathbf{0}_{n-1}\|_n \leq \sum_{i,j} \|a_{i,j}\|_1$$

Let $a_k = [a_{i,j,k}] \in M_n(A)$ be such a sequence, then given an $\varepsilon > 0$ we have a $P \in \mathbb{N}$, such that for $k, k' > P$ we have the bound

$$\|a_k - a_{k'}\|_n = \|[a_{i,j,k}] - [a_{i,j,k'}]\|_n \leq \sum_{i,j} \|a_{i,j,k} - a_{i,j,k'}\|_1 \leq n^2\varepsilon$$

□

Proof of Proposition 2.25. Given a y in B there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = y$, we define $\bar{f}y = \lim_{n\to\infty} f y_n$. To prove that this indeed well-defined let $\{y_n\}_{n\in\mathbb{N}}$ and $\{y'_n\}_{n\in\mathbb{N}}$ that both converge to $y \in B$. We have that

$$\lim_{n\to\infty}\|y_n - y'_n\| \leq \lim_{n\to\infty}\|y_n - y\| + \|y - y'_n\| = 0$$

consequently

$$\lim_{n\to\infty}\|fy_n - fy'_n\| \leq \lim_{n\to\infty}\|f\|\|y_n - y'_n\| = 0$$

and the sequences $\{fy_n\}_{n\in\mathbb{N}}$ and $\{fy'_n\}_{n\in\mathbb{N}}$ converge to the same limit. To prove that this is indeed completely bounded let $y \in M_n(B)$, then $y = [y_{i,j}] = \lim_{k\to\infty}[y_{i,j,k}]$. We have that:

$$\begin{aligned} \|\bar{f}_n y\|_n &= \|\lim_{k\to\infty}[f(y_{i,j,k})]\|_n \\ &\leq \lim_{k\to\infty}\|[f(y_{i,j,k})]\|_n \quad \text{(by Lemma 2.26)} \\ &\leq \lim_{k\to\infty}\|f_n\|_{cb}\|y_k\|_n \quad \text{(by definition of the cb-norm)} \\ &\leq \|f_n\|_{cb} \lim_{k\to\infty}\|y_k\|_n \\ &= \|f_n\|_{cb}\|y\|_n \end{aligned}$$

We can deduce from this that $\|\bar{f}\|_{\text{cb}} \leq \|f\|_{\text{cb}}$ and as $\bar{f}_n x = f_n x$ for all $x \in M_n(A)$ we must have that $\|\bar{f}\|_{\text{cb}} = \|f\|_{\text{cb}}$.

Uniqueness follows from the fact that any contraction is continuous, and the uniqueness of limits. □

When proving that a completely contractive map is an isomorphism, it might sometimes be easier to work with the dual spaces as they have a very concrete description, given that they are cb-normed. The following proposition enables us to do just that.

Proposition 2.27. *[Ble16, Exercise 1.2.9] Let A and B are operator space normed vector spaces, and $f : A \to B$ is an isomorphism of the underlying vector spaces. If $f^* : B^* \to A^*$ is a well-defined, completely isometric and surjective, then f must be completely isometric as well.*

Proof. Given an $a \in M_n(A)$ we have the following sequence of equalities

$$\begin{aligned} \|f_n(a)\| &= \sup\{\|gf_n(a)\| \mid g \in M_n(B^*)\ \|g\| \leq 1\} && \text{(as the double dual embedding is isometric)} \\ &= \sup\{\|gf(a)\| \mid g \in \mathcal{CB}(B, M_n)\ \|gf\| \leq 1\} && \text{(by assumption about complete isometry)} \\ &= \sup\{\|h(a)\| \mid h \in \mathcal{CB}(A, M_n)\ \|h\| \leq 1\} && \text{(by assumption on surjectivity)} \\ &= \sup\{\|h(a)\| \mid h \in M_n(A^*)\ \|h\| \leq 1\} \\ &= \|a\| && \text{(as the double dual embedding is isometric)} \end{aligned}$$

□

2.2 Constructions on operator spaces

We give a brief overview of some common constructions considered on operator spaces, as well as some results about these that will prove useful for Section 3 and Section 4. Many of the constructions parallel constructions considered on Banach spaces, we can define *projective* and *injective* tensor norms of operator spaces with similar properties as the ones on Banach spaces, as well as direct sums with l_p-norms. We will also recall a tensor norm which is not usually considered in Banach space theory, the *Haagerup* tensor product.

As we are interested in describing properties of the category of certain operator spaces, our main focus will be on properties with a categorical flavour, such as functoriality of the constructions, universal properties and the naturality of certain morphisms.

2.2.1 Projective tensor

The projective tensor is for jointly completely bounded maps what the algebraic tensor is for bilinear maps, that is, it characterizes uniquely jointly completely bounded maps. We give several equivalent ways to define this tensor product, ranging from concrete descriptions of the matrix norm to ones induced by embedding the algebraic tensor into an operator space.

Definition 2.28. [ER00, (7.1.8)] Let A and B be operator spaces, then given a $u \in \mathbb{M}_n(A \otimes B)$ we define the *projective norm* on u to be

$$\|u\|_\wedge = \inf\{\|\alpha\|\|a\|\|b\|\|\beta\| \mid \alpha(a \otimes b)\beta = u\}$$

for $a \in M_p(A)$, $b \in M_q(B)$, $\alpha \in M_{n,p\times q}$ and $\beta \in M_{p\times q,n}$

Definition 2.29. [ER00, p. 124] Let A and B be two operator spaces then the *projective tensor product*, denoted $A \widehat{\otimes} B$, is defined as the completion of $A \otimes B$ with respect to $\| - \|_\wedge$.

Proposition 2.30. *[ER00, p. 7.1.1] Let A and B be two operator spaces $A \widehat{\otimes} B$ with operator norm given by $\| - \|_\wedge$ is an operator space.*

The projective tensor is functorial both with respect to completely bounded maps and completely contractive maps.

Proposition 2.31. *[Ble16, p. 68] Given operator spaces A_1, A_2, B_1 and B_2 and completely bounded maps $f_i : A_i \to B_i$, we have a completely bounded map defined by the following assignment on elementary tensors*

$$\begin{array}{rcl} f_1 \widehat{\otimes} f_2 : A_1 \widehat{\otimes} A_2 & \to & B_1 \widehat{\otimes} B_2 \\ a_1 \otimes a_2 & \mapsto & f_1(a_1) \otimes f_2(a_2) \end{array}$$

This assignment makes the projective tensor product functorial

Corollary 2.32. *[ER00, Cor. 7.1.3] Given operator spaces A_1, A_2, B_1 and B_2 and completely contractive maps $f_i : A_i \to B_i$, we have a completely contractive map*

$$f_1 \widehat{\otimes} f_2 : A_1 \widehat{\otimes} A_2 \to B_1 \widehat{\otimes} B_2$$

Now, let $\mathcal{JCB}(A, B; C)$ denote the operator space of jointly completely bounded maps with the cb-norm, using the identification $M_n(\mathcal{JCB}(A, B; C)) \cong \mathcal{JCB}(A, B; M_n(C))$ for the matrix norm.

Proposition 2.33. *[ER00, Prop. 7.1.2] We have a natural isometric isomorphism*

$$\mathcal{CB}(A \widehat{\otimes} B, C) \cong \mathcal{JCB}(A, B; C)$$

Proof. Given a jointly completely bounded map $\varphi : A \times B \to C$, this induces a completely bounded map $\bar{\varphi} : A \widehat{\otimes} B \to C$ by mapping $u = \alpha(a \otimes b)\beta \in M_n(A \widehat{\otimes} B)$ for $a \in M_p(A)$ and $b \in M_q(B)$ to $\alpha(\varphi_{p,q}(a, b))\beta$. We have that

$$\|\bar{\varphi}(u)\|_{\mathrm{cb}} \leq \|\alpha\|\|\varphi_{p,q}\|\|a\|\|b\|\|\beta\| \leq \|\varphi\|_{\mathrm{cb}}\|\alpha\|\|a\|\|b\|\|\beta\|$$

thus, we have that $\|\bar{\varphi}_n(u)\| \leq \|\varphi\|_{\mathrm{cb}}\|u\|_\wedge$ for all $u \in M_n(A \otimes B)$ and consequently $\|\bar{\varphi}\|_{\mathrm{cb}} \leq \|\varphi\|_{\mathrm{cb}}$. To prove the equality let $\varepsilon > 0$, for each such ε we can find a pair $a \in M_p(A)_{\|-\|\leq 1}$ and $b \in M_n(B)_{\|-\|\leq 1}$ such that

$$\|\varphi\|_{\mathrm{cb}} \leq \|\varphi_{p,q}(a, b)\| + \varepsilon$$

As we for any such a and b have that

$$\|a \otimes b\|_\wedge \leq \|a\|\|b\| \leq 1$$

consequently

$$\|\varphi\|_{\text{cb}} \leq \|\bar{\varphi}_{p\times q}(a \otimes b)\| + \varepsilon \leq \|\varphi\|_{\text{cb}} + \varepsilon$$

the equality $\|\bar{\varphi}\|_{\text{cb}} = \|\varphi\|_{\text{cb}}$ must thus hold. □

As a jointly completely bounded map $\varphi : A \times B \to C$ for each $a \in A$ we have a $\varphi(a, -) \in \mathcal{CB}(B, C)$ (and similarly for each $b \in B$) we have that the following proposition holds.

Proposition 2.34. *[ER00, Prop. 7.1.2] Given operator spaces A, B and C we have a natural isomorphism*

$$\mathcal{JCB}(A, B; C) \cong \mathcal{CB}(A, \mathcal{CB}(B, C))$$

Proof. Given a $\varphi \in \mathcal{JCB}(A, B; C)$ this induces a completely bounded map $\tilde{\varphi} : A \to \mathbf{B}(B, C)$ that maps $a \in A$ to $\varphi(a, -)$, we need to prove that $\tilde{\varphi}(a)$ is completely bounded. We have that for any $b \in M_n(B)$

$$\|\tilde{\varphi}(a)_n(b)\| = \|\varphi_{1,n}(a, b)\| \leq \|\varphi\|_{\text{cb}}\|a\|\|b\|$$

consequently $\|\tilde{\varphi}(a)\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}}\|a\|$ and $\tilde{\varphi}(A) \subseteq \mathcal{CB}(B, C)$. Now $\|\tilde{\varphi}\|_{\text{cb}} = \|\varphi\|_{\text{cb}}$ as

$$(\tilde{\varphi}_p(a))_q(b) = [\tilde{\varphi}(a_{i,j})(b_{l,k})] = [\varphi(a_{i,j}, b_{l,k})] = \varphi_{p,q}(a, b)$$

the map $\theta : \mathcal{JCB}(A, B; C) \cong \mathcal{CB}(A, \mathcal{CB}(B, C)) : \varphi \mapsto \tilde{\varphi}$ is an isometry and as the following diagram commutes it must be a complete isometry.

$$\begin{array}{ccc}
M_n(\mathcal{JCB}(A, B; C)) & \xrightarrow{\theta_n} & M_n(\mathcal{CB}(A, \mathcal{CB}(B, C))) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{JCB}(A, B; M_n(C)) & \xrightarrow[\theta]{} & \mathcal{CB}(A, \mathcal{CB}(B, M_n(C)))
\end{array}$$

□

Combining the results from above we deduce the following,

Proposition 2.35. *There is a natural completely isometric isomorphism $\tilde{(-)} : \mathcal{CB}(A \widehat{\otimes} B, C) \cong \mathcal{CB}(A, \mathcal{CB}(B, C))$*

By instantiating the proposition above with the dualizing object $\mathbb{C}$, we get two convenient descriptions of the operator space dual of a projective tensor product.

Corollary 2.36. *There are two completely isometric isomorphisms $(A \widehat{\otimes} B)^* \cong \mathcal{CB}(A, B^*)$ and $(A \widehat{\otimes} B)^* \cong \mathcal{CB}(B, A^*)$.*

As indicated by the corollary above we can now prove that the projective tensor product is commutative.

Proposition 2.37. *[ER00, Prop. 7.1.4] Given operator spaces A and B, there is an isometric isomorphism,*

$$\sigma_{A,B} : A \widehat{\otimes} B \cong B \widehat{\otimes} A$$

natural in both arguments.

Proof. The underlying isomorphism of Banach spaces is generated by the morphism that sends $a \otimes b \to b \otimes a$, note that we have yet to show that this is a (complete) isometry. By Proposition 2.27 it is sufficient to prove that $(\sigma_{A,B})^*$ gives a well-defined surjective isometry of the duals. Upon inspecting how the isomorphisms in Corollary 2.36 would be constructed (see proofs of Proposition 2.33 and Proposition 2.34), we can see that the isomorphism of the duals indeed is given by precomposing with $\sigma_{A,B}$

$$\begin{array}{ccccc} (A \widehat{\otimes} B)^* & \cong & \mathcal{CB}(B, A^*) & \cong & (B \widehat{\otimes} A)^* \\ f & \mapsto & (b \mapsto (f(- \otimes b))) & \mapsto & ((b \otimes a) \mapsto f(a \otimes b)) \end{array}$$

Naturality follows by diagram chase on the dense subspace $A \otimes B$. □

As expected we can also prove associativity and and provide a unit, note in particular the naturality of the isomorphisms provided below.

Proposition 2.38. *[ER00, Prop. 7.1.4] Given operator spaces A, B and C, there is an isometric isomorphism, natural in all three arguments*

$$\alpha_{A,B,C} : (A \widehat{\otimes} B) \widehat{\otimes} C \cong A \widehat{\otimes} (B \widehat{\otimes} C).$$

Proof. The map $\alpha_{A,B,C}$ is defined by

$$\begin{array}{rcl} \alpha_{A,B,C} : (A \widehat{\otimes} B) \widehat{\otimes} C & \cong & A \widehat{\otimes} (B \widehat{\otimes} C) \\ (a \otimes b) \otimes c & \mapsto & a \otimes (b \otimes c) \end{array}$$

This defines an isomorphism on the underlying Banach spaces. By Proposition 2.27 it is then sufficient to prove that the induced map on the dual spaces gives a surjective complete isometry.

By repeated use of Corollary 2.36 and Proposition 2.35 we have the following sequence of isomorphisms

$$((A \widehat{\otimes} B) \widehat{\otimes} C)^* \cong \mathcal{CB}((A \widehat{\otimes} B), C^*) \cong \mathcal{CB}(A, \mathcal{CB}(B, C^*)) \cong \mathcal{CB}(A, (B \widehat{\otimes} C)^*) \cong (A \widehat{\otimes} (B \widehat{\otimes} C))^*$$

$$\begin{array}{rcl} f & \mapsto & (a \otimes b \mapsto \tilde{f}(a \otimes b, -)) \\ & \mapsto & (a \mapsto (b \mapsto \tilde{\tilde{f}}(a, b, -))) \\ & \mapsto & (a \mapsto (b \otimes c \mapsto \tilde{f}(a, b \otimes c))) \\ & \mapsto & (a \otimes (b \otimes c) \mapsto f((a \otimes b) \otimes c)) \end{array} \qquad \begin{array}{rcl} ((A \widehat{\otimes} B) \widehat{\otimes} C)^* & \cong & \mathcal{CB}((A \widehat{\otimes} B), C^*) \\ & \cong & \mathcal{CB}(A, \mathcal{CB}(B, C^*)) \\ & \cong & \mathcal{CB}(A, (B \widehat{\otimes} C)^*) \\ & \cong & (A \widehat{\otimes} (B \widehat{\otimes} C))^* \end{array}$$

Which coincides with the morphism given by precomposing with $\alpha_{A,B,C}$.

Naturality holds on the restriction to the dense subset $(A \otimes B) \otimes C$, and as any dense morphism extends uniquely by Proposition 2.25, α_{-} must consequently be natural. □

Proposition 2.39. *We have isometric isomorphisms*

$$\lambda_A : \mathbb{C} \widehat{\otimes} A \to A$$

and

$$\rho_A : A \to A \widehat{\otimes} \mathbb{C}$$

these are natural in A.

Proof.

$$\begin{array}{rcl} \lambda_A : \mathbb{C} \mathbin{\widehat{\otimes}} A & \to & A \\ n \otimes a & \mapsto & na \end{array}$$

This gives an isomorphism on the underlying vector spaces, to prove that this is indeed an isometry we observe that the following spaces are isometrically isomorphic by λ_A^*

$$\begin{array}{ccccc} A^* & \cong & \mathcal{CB}(A, \mathcal{CB}(\mathbb{C}, \mathbb{C})) & \cong & (\mathbb{C} \mathbin{\widehat{\otimes}} A)^* \\ f & \mapsto & (a \mapsto (n \mapsto nf(a))) & \mapsto & (n \otimes a \mapsto nf(a)) \end{array}$$

Naturality follows again from diagram chasing. We construct ρ_A using the following:

$$\begin{array}{rcl} \rho_A : A & \to & A \mathbin{\widehat{\otimes}} \mathbb{C} \\ a & \mapsto & a \otimes 1 \end{array}$$

To prove that it indeed is a natural isometric isomorphism one can proceed similarly as for λ_{-}. ☐

Another way to define the projective tensor product is via the subspace inclusion $A \otimes B \to (\mathcal{CB}(A, B^*))^*$, identifying $a \otimes b$ with $(f \mapsto f(a)(b))$, and then completing $A \otimes B$ with respect to the induced subspace norm. This characterization is equivalent to the concrete description above as witnessed by the following proposition.

Proposition 2.40. *The inclusion $A \mathbin{\widehat{\otimes}} B \to (\mathcal{CB}(A, B^*))^*$ is a complete isometry.*

Proof. By combining Proposition 2.20 and Corollary 2.36 have an isometric embedding $A \mathbin{\widehat{\otimes}} B \to (A \mathbin{\widehat{\otimes}} B)^{**} \cong (\mathcal{CB}(A, B^*))^*$, which corresponds to the map $a \otimes b \mapsto (f \mapsto f(a)(b))$. ☐

2.2.2 Injective tensor

To continue our Banach Space analogy, we also define an injective norm on the tensor product. Again we can define this tensor in several equivalent ways, either by giving a norm explicitly or by embedding it into another operator space, we present both below.

Definition 2.41. [ER00, (8.1.7)] Let A and B be operator spaces, given a $u \in \mathbb{M}_n(A \otimes B)$ we defined the *injective norm* on u to be

$$\|u\|_\vee = \sup\{\|(f \otimes g)_n(u)\| \mid f \in M_p(A^*), g \in M_q(B^*), \|f\|, \|g\| \leq 1\}$$

Definition 2.42. [ER00, p. 139] Let A and B be two operator spaces then the *injective tensor product*, denoted $A \mathbin{\check{\otimes}} B$, is defined as the completion of $A \otimes B$ with respect to $\| - \|_\vee$.

Proposition 2.43. *Let A and B be two operator spaces $A \mathbin{\check{\otimes}} B$ with operator norm given by $\| - \|_\vee$ is an operator space.*

We can equivalently consider the following realization,

Proposition 2.44. *[ER00, Prop. 8.1.1] Let A and B be operator spaces then the injective norm is equivalent to the norm induced by the embedding:*

$$\begin{array}{rcl} A \otimes B & \to & \mathcal{CB}(A^*, B) \\ a \otimes b & \mapsto & (f \mapsto f(a)b) \end{array}$$

The following is a useful variation of the previous result.

Proposition 2.45. *[ER00, Prop. 8.1.2] Let A and B be operator spaces the following is a completely isometric embedding:*

$$\begin{array}{rcl} A^* \check{\otimes} B & \to & \mathcal{CB}(A, B) \\ f \otimes b & \mapsto & (a \mapsto f(a)b) \end{array}$$

Restricting to finite dimensional spaces the injective tensor has a particularly nice description.

Proposition 2.46. *Given two finite dimensional operator spaces A and B the isometric embedding*

$$A \check{\otimes} B \to \mathcal{CB}(A^*, B)$$

is an isometric isomorphism.

Proof. The argument follows by combining Proposition 2.45, Corollary 2.22 and the rank-nullity argument on the underlying vector spaces. □

Proposition 2.47. *[ER00, Cor. 8.1.7] The injective tensor product is naturally commutative and associative.*

2.2.3 Haagerup tensor

In this section we introduce the Haagerup tensor norm, which gives a tensor product with some particular properties.

Definition 2.48. [ER00, (9.1.8)] Given $a \in \mathbb{M}_{n,r}(A)$ and $b \in \mathbb{M}_{r,m}(B)$ the *matrix inner product* (denoted $\odot$) of a and b is defined as

$$(a \odot b)_{i,j} = \sum_{k=1}^{r} a_{i,k} \otimes b_{k,j}$$

Definition 2.49. [ER00, (9.2.1)] Given an $v \in \mathbb{M}_n(A \otimes B)$ the *Haagerup norm* of v is defined as

$$\|v\|_h = \inf\{\|u\|\|w\| \mid u \odot w = v\}$$

Using the Haagerup norm we can define yet another type of tensor product on operator spaces.

Definition 2.50. [ER00, p. 153] Let A and B be two operator spaces then the *Haagerup tensor product*, denoted $A \otimes_h B$, is defined as the completion of $A \otimes B$ with respect to $\| - \|_h$.

Proposition 2.51. *[ER00, Th. 9.2.1] Let A and B be two operator spaces $A \otimes_h B$ with operator norm given by $\| - \|_h$ is an operator space.*

Proposition 2.52. *[Ble16, par. 3.1.13] Given operator spaces A_1, A_2, B_1 and B_2 and completely bounded maps $f_i : A_i \to B_i$, we have a completely bounded map defined by the following assignment on elementary tensors*

$$\begin{array}{rcl} f_1 \otimes_h f_2 : A_1 \otimes_h A_2 & \to & B_1 \otimes_h B_2 \\ a_1 \otimes a_2 & \mapsto & f_1(a_1) \otimes f_2(a_2) \end{array}$$

This assignment makes the projective tensor product functorial

Corollary 2.53. *[ER00, Prop. 9.2.5] Given operator spaces A_1, A_2, B_1 and B_2 and completely contractive maps $f_i : A_i \to B_i$, we have a completely contractive map*

$$f_1 \otimes_h f_2 : A_1 \otimes_h A_2 \to B_1 \otimes_h B_2$$

Proposition 2.54. *[ER00, Proposition 9.3.2] Let A be an operator space and H be a Hilbert space then we have isometric isomorphisms*

$$A \otimes_h H_c \cong A \widehat{\otimes} H_c$$

$$H_c \otimes_h A \cong H_c \check{\otimes} A$$

It is often hard to compute the Haagerup tensor product of two operator spaces explicitly, by the proposition above we can provide the following two concrete examples.

Example 2.55. Given a Hilbert space H we have the following isomorphisms for the Haagerup tensor of the Hilbert column operator space H_c

$$\begin{array}{rcl} H_c \otimes_h (H_c)^* & \cong & H_c \widehat{\otimes} (H_c)^* \\ (H_c)^* \otimes_h H_c & \cong & (H_c)^* \check{\otimes} H_c \end{array}$$

Note that this example also witnesses the non-commutativity of this tensor.

While the Haagerup tensor is non-commutative it still have associativity and a unit, note again that the isomorphisms below are natural in each argument.

Proposition 2.56. *[ER00, Prop. 9.2.7] Given operator spaces A, B and C, there is a completely isometric isomorphism, natural in all three arguments*

$$\begin{array}{rcl} \alpha'_{A,B,C} : (A \otimes_h B) \otimes_h C & \cong & A \otimes_h (B \otimes_h C) \\ (a \otimes b) \otimes c & \mapsto & a \otimes (b \otimes c) \end{array}$$

Proposition 2.57. *We have isometric isomorphisms*

$$\lambda'_A : \mathbb{C} \otimes_h A \to A$$

and

$$\rho'_A : A \to A \otimes_h \mathbb{C}$$

these are natural in A.

One of the particular properties the Haagerup tensor product has is that it is dual to itself, as illustrated by the propositions below.

Proposition 2.58. *[ER00, Th. 9.4.7] Given two operator spaces A and B there is a natural isometric embedding*

$$\begin{array}{rcl} A^* \otimes_h B^* & \to & (A \otimes_h B)^* \\ f \otimes g & \mapsto & (a \otimes b \mapsto f(a)g(b)) \end{array}$$

Corollary 2.59. *[ER00, Cor. 9.4.8] If either A or B is finite dimensional then the previous embedding is an isometric isomorphism.*

2.2.4 Direct sums

To revert back into our Banach space analogy, we can also define various norms on the direct sum of two operator spaces. The two most interesting ones, the 1- and ∞-norm turn out to coincide with a familiar construction in category theory, a fact we will address in the next section.

Definition 2.60. [Ble16, par. 1.2.17] Let A and B be operator spaces, the ∞-*direct sum* of these operator spaces has as underlying normed space the cartesian product, $A \times B$ equipped with the ∞ norm, $\|(a,b)\|_\infty := \max(\|a\|, \|b\|)$. The operator space structure is then given by the following isometric identifications for each matrix space.

$$M_n(A \oplus^\infty B) \cong M_n(A) \oplus^\infty M_n(B)$$

Concretely, the operator space ∞-norm can be described as

Proposition 2.61. *Given* $u = [(a_{i,j}, b_{i,j})] \in M_n(A \oplus^\infty B)$

$$\|u\|_n = max(\|[a_{i,j}]\|, \|[b_{i,j}]\|)$$

Given this presentation of the norm it is that the underlying Banach space of the operator space $A \oplus^\infty B$ coincides with the ℓ_∞ direct sum of the underlying Banach spaces of A and B

Definition 2.62. [Ble16, par. 1.3.14] Let A and B be operator spaces, the 1-*direct sum* of these operator spaces has as underlying normed space the cartesian product, $A \times B$, with operator space structure induced by the identification of $A \oplus^1 B$ included into $(A^* \oplus^\infty B^*)^*$, by the map $(a,b) \mapsto ((f,g) \mapsto f(a) + g(b))$.

Concretely, the operator space 1-norm can be described as the following:

Proposition 2.63. *Given* $u = [(a_{i,j}, b_{i,j})] \in M_n(A \oplus^1 B)$

$$\|u\|_n = sup\{\|[f_{i,j}(a_{i,j}) + g_{i,j}(b_{i,j})]\| \mid (f,g) \in M_n(A^* \oplus^\infty B^*),\ \|(f,g)\| \leq 1\}$$

Note that the underlying Banach space of the operator space $A \oplus^1 B$ coincides with the ℓ_1 direct sum of the underlying Banach spaces of A and B.

3 MALL

3.1 Models of MALL

The fundamental idea in categorical proof theory is that propositions should be interpreted as objects and proofs should be interpreted as morphisms. Below we give a brief overview of the notions required for a categorical model of classical *MALL*, and the interpretations employed.

Definition 3.1 ($*$-autonomous category)**.** A symmetric monoidal closed category is called $*$-*autonomous* if the transpose $\partial_A : A \to ((A \multimap \bot) \multimap \bot)$ of the evaluation map $\mathrm{eval}_A : A \otimes (A \multimap \bot) \to \bot$ is an isomorphism.

We can see the natural isomorphism ∂_- as the statement that the double negation of a formula is equivalent to the formula itself in classical logic, we will often denote $A \multimap \bot$ by A^*.

Proposition 3.2. *[Mel09, Ch. 5] Any $*$-autonomous category is a model of classical MALL where* $[\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!]$ *and* $[\![A \parr B]\!] := [\![A]\!]^* \multimap [\![B]\!]$.

Upon inspection of the formation rules of the additive fragment of linear logic it is clear that we only need our $*$-autonomous category to admit finite products and coproducts in order to model MALL.

Proposition 3.3. *[Mel09, Ch. 5] Any cartesian $*$-autonomous category (hence also cocartesian) is a model of classical MALL logic where* $[\![A \,\&\, B]\!] := [\![A]\!] \times [\![B]\!]$ *and* $[\![A \oplus B]\!] := [\![A]\!] + [\![B]\!]$.

The class of cartesian $*$-autonomous categories is sound and complete with respect to MALL, in particular this implies that it is cut-elimination invariant, see [Mel09] for a comprehensive review of categorical models of linear logic.

3.2 fdOS as model of MALL

We now introduce the category of finite dimensional operator spaces, which we will proceed to verify is a model of classical MALL.

Definition 3.4. Let ***fdOS*** denote the category of abstract finite dimensional operator spaces and completely contractive maps.

In Section 2.2.1 we proved that the projective tensor product was functorial with respect to completely contractive maps, Corollary 2.32, as well as the fact that there exists natural isomorphisms shaped like the associator and unitors of a monoidal category, Proposition 2.38 and Proposition 2.39. As expected we have the following monoidal structure on **fdOS**.

Proposition 3.5. ***fdOS*** *equipped with the projective tensor product,* $(\textbf{\textit{fdOS}}, \mathbb{C}, \widehat{\otimes})$*, is monoidal.*

To prove this we first prove a small lemma:

Lemma 3.6. *The forgetful functor* $U : \textbf{\textit{fdOS}} \to \textbf{\textit{Vect}}$ *is faithful and maps the projective tensor product to the algebraic tensor of vector spaces.*

Proof. Faithfulness follows from the fact that any completely contractive map is a linear map. Tensor product preservation follows from the fact that any finite dimensional normed space is complete Proposition 2.2, as the underlying vector space of $A \widehat{\otimes} B$ is $A \otimes B$, for finite dimensional operator spaces A and B. □

Proof of Proposition 3.5. The associator and unitors are given by the natural isomophisms in Proposition 2.38 and Proposition 2.39. The commutativity of the MacLane pentagon and the triangle identity, now follows from Lemma 3.6. □

To prove that **fdOS** is monoidal closed with respect to this monoidal structure it is sufficient to prove that Proposition 2.35 restricts to completely contractive maps.

Lemma 3.7. *There is a natural isomorphism* $\textbf{\textit{fdOS}}(A \widehat{\otimes} B, C) \cong \mathcal{JCC}(A, B; C)$

Proof. Jointly completely contractive maps $A \times B \to C$ correspond exactly to completely contractive $A \widehat{\otimes} B \to C$ by the fact that the isomorphism in Proposition 2.33 is isometric. □

Lemma 3.8. *There is a natural isomorphism* $\theta : \mathcal{JCC}(A, B; C) \cong \textbf{\textit{fdOS}}(A, \mathcal{CB}(B, C))$.

Proof. Let $\varphi : A \times B \to C$ then $\theta(\varphi) : A \to \mathcal{CB}(B, C)$ is defined by $\theta(\varphi)(a)(b) = \varphi(a, b)$. We begin by proving that the image of A under φ indeed is in $\mathcal{CB}(B, C)$. Let thus $a \in A$ and $b \in M_n B$ for some $n \in \mathbb{N}$, then

$$\|\theta(\varphi)(a)_n(b)\| = \|\varphi_{1;n}(a, b)\| \leq \|\varphi\|_{cb}\|a\|\|b\|$$

Consequently $\|\theta(\varphi)(a)\|_{cb} \leq \|\varphi\|_{cb}\|a\|$, i.e. $\theta(\varphi)(a)$ is completely bounded. Now we prove that it is indeed an isomorphism by constructing an inverse, given instead a $\psi : A \to \mathcal{CB}(B, C)$ let θ^{-1} be defined by $\theta^{-1}(\psi)(a, b) = (\psi)(a)(b)$. We verify that $\theta^{-1}(\psi)$ indeed is completely contractive:

$$\|\theta^{-1}(\psi)_{p;q}(a, b)\| = \|\psi_p(a)_q(b)\| \leq \|\psi_q(a)\|_{cb}\|b\| \leq \|a\|\|b\|$$

The last inequality is true as ψ is completely contractive. It is clear that $\theta\theta^{-1}(\psi) = \psi$ and $\theta^{-1}\theta(\varphi) = \varphi$. □

This is essentially saying that $\varphi : A \times B \to C$ is jointly completely contractive if and only if $\theta(\varphi) : A \to \mathcal{CB}(B, C)$ is a complete contraction. Combining the two lemmas then gives us the monoidal closure.

Proposition 3.9. ***fdOS*** *with the projective tensor product* $\widehat{\otimes}$ *is monoidal closed.*

Proof. Combining the previous two lemmas we have a natural isomorphism $\mathbf{fdOS}(A \widehat{\otimes} B, C) \cong \mathbf{fdOS}(A, \mathcal{CB}(B, C))$. That is, the internal hom of **fdOS** is thus $\mathcal{CB}(-, -)$. □

Now we leverage the fact the we only consider finite dimensional spaces to get a $*$-autonomous structure.

Proposition 3.10. *The category* ***fdOS*** *is* $*$*-autonomous, with* $\mathbb{C}$ *as dualizing object (or operator space dual* $(-)^* = \mathcal{CB}(-, \mathbb{C})$ *as dualization functor).*

Proof. The transpose of the evaluation map $\mathrm{ev}_{A,\mathbb{C}} : \mathcal{CB}(A, \mathbb{C}) \otimes A \to \mathbb{C}$ is the canonical embedding of A into its double dual

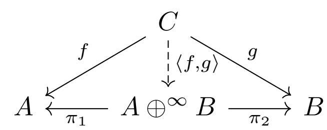
$$\begin{array}{rcl} d_A : A & \to & \mathcal{CB}(\mathcal{CB}(A, \mathbb{C}), \mathbb{C}) \\ a & \mapsto & (f \mapsto f(a)) \end{array}$$

consequently the claim follows by Corollary 2.22 □

We have now proved that **fdOS** is a model of MLL, with the tensor $\otimes$ given by $\widehat{\otimes}$ and the par $⅋$ given by $\mathcal{CB}((-)^*, -)$, which we in Proposition 2.46 proved is isomorphic to the injective tensor product, $\check{\otimes}$. Consequently, the multiplicative connectives can be modeled by constructions from operator space theory. Note that this is a non-degenerate model of MLL as the tensor and the parr do not coincide. To get a model of MALL we also need additive structure, to this end consider the two direct sums defined in Section 2.2.4, we will prove that these coincide with the product and coproducts in **fdOS**.

Proposition 3.11. $\oplus^\infty$ *is the product in* ***fdOS****.*

Proof. Let A, B, C be some operator spaces and $f : C \to A$ and $g : C \to B$ be completely contractive maps, then we get a unique map from C to $A \oplus^\infty B$ given by the pair:

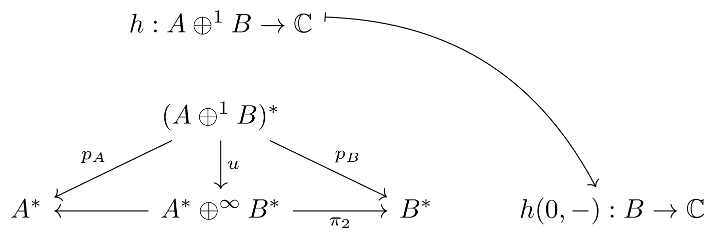


The map $\langle f, g\rangle$ is completely contractive and the $\oplus^\infty$ is the product in **fdOS**. □

We can prove that $\oplus^1$ is the dual to $\oplus^\infty$ under the operator space dual.

Proposition 3.12. *The ℓ_1 direct sum, $\oplus^1$, is dual to $\oplus^\infty$ under the operator space dual, that is, there are natural completely isometric isomorphisms $(A \oplus^1 B)^* \cong A^* \otimes^\infty B^*$ and $(A \oplus^\infty B)^* \cong A^* \otimes^1 B^*$.*

Proof. Consider the morphism given by the universal property of the product in the following diagram:



The other leg is defined similarly, $p_A := h(-, 0)$.

Both p_A and p_B are both completely contractive, we show this for p_B. Let $h \in M_n((A \oplus^1 B)^*) \cong \mathcal{CB}(A \oplus^1 B, M_n)$ then:

$$\|h(0, b)\| \leq \|h\|_{\text{cb}}(\|b\| + \|0\|) = \|h\|_{\text{cb}}\|b\|$$

thus $\|p_b(h)\|_{\text{cb}} \leq \|h\|_{\text{cb}}$. Consequently u is completely contractive. As u has an inverse, $v : A^* \oplus^\infty B^* \to (A \oplus^1 B)^*$, defined by $v(f, g)(a, b) = f(a) + g(b)$ that is also completely contractive. To prove this let $(f, g) \in M_n(A^* \oplus^\infty B^*) \cong \mathcal{CB}(A, M_n) \oplus^\infty \mathcal{CB}(B, M_n)$ We have that:

$$\|v_n(f, g)(a, b)\| = \|f(a)+g(b)\| \leq \|f(a)\|+\|g(b)\| \leq \|f\|_{\text{cb}}\|a\|+\|g\|_{\text{cb}}\|b\| \leq \max(\|f\|_{\text{cb}}, \|g\|_{\text{cb}}\|)(\|a\|+\|b\|)$$

consequently $\|v_n(f, g)\| \leq \max(\|f\|_{\text{cb}}, \|g\|_{\text{cb}})$ for all $n \in \mathbb{N}$.

As both u and v are completely contractive by Proposition 2.24, u is a completely isometric isomorphisms and $(A \oplus^1 B)^* \cong A^* \oplus^\infty B^*$.

We can construct the other isomorphism similarly, but by the $*$-autonomous structure of **fdOS** it follows from the isomorphisms above. □

Corollary 3.13. *The ℓ_1 direct sum, $\oplus^1$ is the coproduct in **fdOS**.*

It now follows that we can build morphism in **fdOS** that correspond to derivations in MALL. For example, we can easily construct isomorphisms corresponding to the distibutivity laws in MALL.

Proposition 3.14. *There is a natural completely isometric isomorphism $(A \check{\otimes} (B \oplus^\infty C)) \cong (A \check{\otimes} B) \oplus^\infty (A \check{\otimes} C)$.*

Proof. We make use of Proposition 2.46 and the isomorphism is thus $\mathcal{CB}(A^*, (B \oplus^\infty C)) \cong \mathcal{CB}(A^*, B) \oplus^\infty \mathcal{CB}(A^*, C)$. We prove something stronger: $\mathcal{CB}(A, (B \oplus^\infty C)) \cong \mathcal{CB}(A, B) \oplus^\infty \mathcal{CB}(A, C)$. The underlying isomorphism comes from the universal property of the product and is thus natural. We start by proving that this is an isometry, let $f_n : M_n(A) \to M_n(B)$ and $g_n : M_n(A) \to M_n(C)$ and let $a \in M_n(A)$, then

$$\|\langle f_n, g_n\rangle(a)\| = (f_n(a), g_n(a)) = \max(\|f_n(a)\|, \|g_n(a)\|) \leq \max(\|f\|_{\text{cb}}, \|g\|_{\text{cb}})\|a\|$$

thus $\|\langle f, g\rangle\|_{\text{cb}} \leq \|(f, g)\|_{\text{cb}}$. At the same time we have

$$\|f_n\| \leq \max(\|f_n(a)\|, \|g_n(a)\|) = \|(f_n(a), g_n(a))\| \leq \|\langle f_n, g_n\rangle\|\|a\| \leq \|\langle f, g\rangle\|_{\text{cb}}\|a\|$$

thus $\|\langle f, g\rangle\|_{\text{cb}} = \|(f, g)\|_{\text{cb}}$. □

By the $*$-autonomous structure on **fdOS** we also have the other distibutivity law as a corollary.

Corollary 3.15. *There is a natural completely isometric isomorphism* $(A \widehat{\otimes} (B \oplus^1 C)) \cong (A \widehat{\otimes} B) \oplus^1 (A \widehat{\otimes} C)$.

4 BV-structure

4.1 BV-logic

Guglielmi presented in [Gug99] an extension of the multiplicative fragment of linear logic by a non-commutative, self-dual connective, referred to as the *sequantial*, this extension is called *Basic System V* or BV-logic for short. BV-logic does not have an associated sequent style calculus, this turns out to be impossible, see [Gug99]. Instead the deductive system is a calculus of structures style system, which has a distinctive feature, *deep inference*, this means that inference rules can be applied anywhere in the context. From a categorical proof theory perspective having deep inference indicates that the context formation rules should be interpreted by endofunctors on the categorical model.

4.2 BV-category

A BV-category is a category equipped with monoidal structures that satisfy the coherence conditions of BV-logic, in which, it is possible to encode its structural rules as natural maps between propositions. The following formal definition is equivalent to the one first presented in [BPS10].

Definition 4.1. [BPS10, Sec. 3] A symmetric linear distributive category $(\mathcal{C}, \bot, \otimes, \parr)$ with an additional monoidal structure $(\mathcal{C}, I, \rhd)$ that is normal duoidal to $(\mathcal{C}, \bot, \otimes)$ is called a *BV-category*

For $(\mathcal{C}, I, \rhd)$ to be normal duoidal to $(\mathcal{C}, \bot, \otimes)$ we have natural transformations:

$$w : (A \rhd B) \otimes (C \rhd D) \to (A \otimes C) \rhd (B \otimes D)$$

and

$$i : \bot \cong I \qquad \delta_I : \bot \to \bot \rhd \bot \qquad \mu_I : I \otimes I \to I$$

the morphism w is often called *weak interchange*.

Along with coherence conditions for associativity,

$$
\begin{array}{ccc}
((A \rhd B) \otimes (C \rhd D)) \otimes (E \rhd F) & \xrightarrow{\alpha} & (A \rhd B) \otimes ((C \rhd D) \otimes (E \rhd F)) \\
\downarrow & & \downarrow \\
((A \otimes C) \rhd (B \otimes D)) \otimes (E \rhd F) & & (A \rhd B) \otimes ((C \otimes E) \rhd (D \rhd F)) \\
\downarrow & & \downarrow \\
((A \otimes C) \otimes E) \rhd ((B \otimes D) \otimes F) & \xrightarrow[\alpha \rhd \alpha]{} & (A \otimes (C \otimes E)) \rhd (B \otimes (D \otimes F))
\end{array}
$$

$$
\begin{array}{ccc}
((A \rhd C) \rhd E) \otimes ((B \rhd D) \rhd F) & \xrightarrow{\alpha \otimes \alpha} & (A \rhd (C \rhd E)) \otimes (B \rhd (D \rhd F)) \\
\downarrow & & \downarrow \\
((A \rhd C) \otimes (B \rhd D)) \rhd (E \otimes F) & & (A \otimes B) \rhd ((C \rhd E) \otimes (D \rhd F)) \\
\downarrow & & \downarrow \\
((A \otimes B) \rhd (C \otimes D)) \rhd (E \otimes F) & \xrightarrow[\alpha]{} & (A \otimes B) \rhd ((C \otimes D) \rhd (E \otimes F))
\end{array}
$$

and unitality,

$$
\begin{array}{ccc}
\bot \otimes (A \rhd B) & \xrightarrow{\delta_\bot \otimes \mathrm{id}} & (\bot \rhd \bot) \otimes (A \rhd B) \\
{\scriptstyle \lambda_{A \rhd B}} \uparrow & & \downarrow {\scriptstyle w} \\
A \rhd B & \xrightarrow[\lambda_A \rhd \lambda_B]{} & (\bot \otimes A) \rhd (\bot \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
(A \rhd B) \otimes \bot & \xrightarrow{\delta_\bot \otimes \mathrm{id}} & (A \rhd B) \otimes (\bot \rhd \bot) \\
{\scriptstyle \rho_{A \rhd B}} \uparrow & & \downarrow {\scriptstyle w} \\
A \rhd B & \xrightarrow[\rho_A \rhd \rho_B]{} & (A \otimes \bot) \rhd (B \otimes \bot)
\end{array}
$$

$$
\begin{array}{ccc}
I \rhd (A \otimes B) & \xleftarrow[\mu_I \rhd \mathrm{id}]{} & (I \otimes I) \rhd (A \otimes B) \\
{\scriptstyle \lambda_{A \rhd B}} \uparrow & & {\scriptstyle w} \uparrow \\
A \otimes B & \xrightarrow[\lambda_A \otimes \lambda_B]{} & (I \rhd A) \otimes (I \rhd B)
\end{array}
\qquad
\begin{array}{ccc}
(A \otimes B) \rhd I & \xleftarrow[\mathrm{id} \rhd \mu_I]{} & (A \otimes B) \rhd (I \otimes I) \\
{\scriptstyle \rho_{A \rhd B}} \uparrow & & {\scriptstyle w} \uparrow \\
A \otimes B & \xrightarrow[\rho_A \otimes \rho_B]{} & (A \rhd I) \otimes (B \rhd I)
\end{array}
$$

as well as diagrams saying that $\bot$ is a comonoid in $(\mathcal{C}, I, \rhd)$ and that I is a monoid in $(\mathcal{C}, \bot, \otimes)$.

Note that in a BV-category, the context formation rules are indeed interpreted by endofunctors as any monoidal structure gives two families of endofunctors. To properly describe BV-logic we also need a notion of duality, and as a symmetric linear distributive category equipped with negation is the same as a $*$-autonomous category under the interpretation of the par as $A \parr B := A^* \multimap B$, a BV-category with negation is defined be the following:

Definition 4.2. [BPS10, Sec. 4] A *BV-category with negation* is a $*$-autonomous category $(\mathcal{C}, \bot, \otimes, \multimap)$ with an additional monoidal structure $(\mathcal{C}, J, \rhd)$ that is normal duoidal to $(\mathcal{C}, I, \otimes)$

4.2.1 Interpreting BV-logic derivations?

In a BV-category the coherence conditions and inference rules occur as natural morphisms induced by the structure, consequently we can interpret derivations of formulas as morphisms in our category. For example the self-dualness of $\triangleright$ can be derived in the following way:

Example 4.3. [BPS10, Th. 4.6] The self-duality isomorphism is given by the transpose of

$$(A^* \triangleright B^*) \otimes (B \triangleright A) \xrightarrow{w} (A^* \otimes A) \triangleright (B^* \otimes B) \xrightarrow{\mathrm{eval}_A \triangleright \mathrm{eval}_B} \bot \triangleright \bot \xrightarrow{w^*} \bot$$

BV-logic has a slightly different cut-rule, thus, as noted in [BPS10], the coherence conditions that a categorical model should satisfy are not clear. Hence we do not address soundness and completeness of BV-categories with respect to BV-logic.

4.3 BV-structure of fdOS

As implied in Section 2.2.3 we can define yet another monoidal structure on **fdOS** using the Haagerup tensor on operator spaces. Recall that this tensor product is self-dual on finite operator spaces, as well as non-commutative and it shares the same unit as the projective tensor. We have, in fact, that **fdOS** is an example of a BV-category.

Proposition 4.4. ***fdOS*** *is a BV-category with negation with using the following structures* $(\boldsymbol{fdOS}, \mathbb{C}, \widehat{\otimes}, \mathcal{CB}(-,-))$ *and* $(\boldsymbol{fdOS}, \mathbb{C}, \otimes_h)$*.*

The duoidality follows naturally from the duality of the $*$-autonomous structure and the fact that $\otimes_h$ is self-dual and functorial with respect to completely bounded maps, and the normality follows from $\widehat{\otimes}$ and $\otimes_h$ sharing a unit. We split the formal proof of Proposition 4.4 into smaller steps.

Proposition 4.5. ***fdOS*** *equipped with the Haagerup tensor product is monoidal.*

To prove this we prove a small lemma:

Lemma 4.6. *The forgetful functor* $U : \boldsymbol{fdOS} \to \boldsymbol{Vect}$ *is faithful and maps the Haagerup tensor product to the algebraic tensor of vector spaces.*

Proof. Faithfulness follows from the fact that any completely contractive map is a linear map. Tensor product preservation follows from the fact that any finite dimensional normed space is complete Proposition 2.2, as the underlying vector space of $A \otimes_h B$ is $A \otimes B$, for finite dimensional operator spaces A and B. □

Proof. The associator is given by the morphism Proposition 2.56, and the unitors by Proposition 2.57. The commutativity of the MacLane pentagon and the triangle identity follows from Lemma 4.6. □

Proposition 4.7. *The Haagerup tensor monoidal structure on* ***fdOS****,* $(\boldsymbol{fdOS}, \mathbb{C}, \widehat{\otimes})$*, is normal duoidal to the monoidal category* $(\boldsymbol{fdOS}, \mathbb{C}, \otimes_h)$*.*

Proof. In order to construct a morphism $w : (A \otimes_h B) \widehat{\otimes} (C \otimes_h D) \to (A \widehat{\otimes} C) \otimes_h (B \widehat{\otimes} D)$, we construct a morphism whose dual will be w. It is constructed as the following composition:

$$(A \check{\otimes} B) \otimes_h (C \check{\otimes} D) \cong \mathcal{CB}(A^*, B) \otimes_h \mathcal{CB}(C^*, D) \xrightarrow{(f \otimes g \mapsto f \otimes g)} \mathcal{CB}(A^* \otimes_h C^*, B \otimes_h D) \cong \mathcal{CB}((A \otimes_h C)^*, B \otimes_h D) \cong (A \otimes_h C) \check{\otimes} (B \otimes_h D)$$

Since the Haagerup tensor and the projective tensor share a unit our i is the identity on $\mathbb{C}$, δ is the isomorphism $\mathbb{C} \cong \mathbb{C} \otimes_h \mathbb{C}$ and μ is $\mathbb{C} \widehat{\otimes} \mathbb{C} \cong \mathbb{C}$. Commutativity of the coherence diagrams in Section 4.2 follows from Lemma 3.6 and Lemma 4.6 as they commute in **Vect** for the dual diagrams. □

5 Conclusion

In this work we studied the properties of the category of finite dimensional operator spaces (**fdOS**). In order to be self-contained, we give an introduction to operator space theory as well as an overview of the relevant constructions.

In doing so, we proved that **fdOS** is a cartesian and cocartesian $*$-autonomous category, that is, a model of classical multiplicative additive linear logic. In particular, we proved that the connectives can be interpreted by constructions natural to operator space theory. Indeed, we proved that the multiplicative connectives can be interpreted by the projective and injective tensors on operator spaces, and the additives by the l_∞- and l_1-direct sums. Additionally, we also proved that the Haagerup tensor product has similar properties as the sequential connective in BV-logic, and that this tensor product gives the additional monoidal structure needed for **fdOS** to be a BV-category with negation.

Having shown that **fdOS** is a model of MALL, where the duality functor that we model negation with is the functorial extension of the operator space dual, indicates that we can describe the Heisenberg-Schrödinger duality using a linear logic negation. The next step would be to carve out from **fdOS** a new category whose structure reflects the Heisenberg-Schrödinger duality. This way we can identify the relevant categorical structure in order to be able to find a type system where the linear negation of linear logic describes the Heisenberg-Schrödinger duality.

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